Faithful Teleportation with Arbitrary Pure or Mixed Resource States

Ming-Jing Zhao[1], Zong-Guo Li[2], Shao-Ming Fei[1,3], Zhi-Xi Wang[1], Xianqing Li-Jost[3]

[1] School of Mathematical Sciences, Capital Normal University, Beijing 100048, China

[2] College of Science, Tianjin University of Technology, Tianjin, 300191, China

[3] Max-Planck-Institute for Mathematics in the Sciences, 04103, Leipzig, Germany

Abstract

We study faithful teleportation systematically with arbitrary entangled states as resources. The necessary conditions of mixed states to complete perfect teleportation are proved. Based on these results, the necessary and sufficient conditions of faithful teleportation of an unknown state $|\phi\rangle$ in $\mathbb{C}^d$ with entangled source ρ in $\mathbb{C}^m \otimes \mathbb{C}^d$ and $\mathbb{C}^d \otimes \mathbb{C}^n$ are derived. It is shown that for ρ in $\mathbb{C}^m \otimes \mathbb{C}^d$, ρ must be a maximally entangled state, while for ρ in $\mathbb{C}^d \otimes \mathbb{C}^n$, ρ must be a pure maximally entangled state. Moreover, we show that the sender's measurements must be all projectors of maximally entangled pure states. The relations between the entanglement of formation of the resource states and faithful teleportation are also discussed. PAC numbers:

03.67.Mn, 03.65.Ud

I. INTRODUCTION

Quantum teleportation plays an important role in quantum information processing. It, employing classical communication and shared resource of entanglement, allows to transmit an unknown quantum state from a sender to a receiver that are spatially separated. Bennett *et. al.* [1] first demonstrated the teleportation of an arbitrary qubit state in terms of an entangled Einstein-Podolsky-Rosen pair. Generally it has proved that only maximally entangled pure states in $\mathbb{C}^d \otimes \mathbb{C}^d$ could faithfully teleport an arbitrary pure state in $\mathbb{C}^d$ [2, 3].

Multipartite states have been also used in faithful and deterministic teleportation. For instance, the three-qubit GHZ state and a class of W states can be used for faithful teleportation of one qubit state [4, 5]. In fact, they are all maximally entangled pure states in $\mathbb{C}^2 \otimes \mathbb{C}^2$ between Alice and Bob. Some five-qubit state is capable of faithful teleportation of arbitrary two-qubit states [6]. The tensor products of two Bell states [7] and the genuine four-qubit entangled states [8] are also used

for faithful teleportation of two-qubit states, in which the teleportation channels can be regarded as maximally entangled states in $\mathbb{C}^4 \otimes \mathbb{C}^4$ shared by the sender and receiver. Basically multipartite pure entangled states are analogous to higher dimensional bipartite pure states in quantum teleportation.

In a realistic case, due to the decoherence the pure maximally entangled states may evolve into mixed entangled states, which could make the teleportation of a state in $\mathbb{C}^d$ imperfect, if the shared entangled state is in $\mathbb{C}^d \otimes \mathbb{C}^d$.

In this paper, we study faithful teleportation with high dimensional bipartite pure or mixed states as the resource. We investigate faithful teleportation of an arbitrary pure state $|\phi\rangle$ in $\mathbb{C}^d$ by using entangled state ρ in $\mathbb{C}^m \otimes \mathbb{C}^n$ $(m, n \geq d)$. We first derive the necessary conditions of mixed states as entangled resources to fulfill faithful teleportation. The necessary and sufficient conditions for faithful teleportation are obtained for resource states ρ in $\mathbb{C}^m \otimes \mathbb{C}^d$ and $\mathbb{C}^d \otimes \mathbb{C}^n$. We show that ρ in $\mathbb{C}^m \otimes \mathbb{C}^d$, either pure or mixed, must be maximally entangled. While ρ in $\mathbb{C}^d \otimes \mathbb{C}^n$ must be pure maximally entangled. Moreover, to fulfill faithful teleportation, the sender's measurements must be all projectors of maximally entangled pure states. For ρ in $\mathbb{C}^m \otimes \mathbb{C}^n$, $m,\ n > d$, we present some classes of states for faithful teleportation.

II. TELEPORTATION WITH ENTANGLED RESOURCE STATES IN $\mathbb{C}^m \otimes \mathbb{C}^n$

Let $|\phi\rangle = \sum_{i=1}^{d} \alpha_i |i\rangle$ be the unknown pure state that is to be sent from Alice to Bob, where $\{|i\rangle\}_{i=1}^{d}$ is the orthonormal basis of $\mathbb{C}^d$. Let ρ be the entangled state shared by Alice and Bob. To carry out teleportation Alice needs to perform projective measurements on her two particles: one in state $|\phi\rangle$ and one part of the entangled state ρ. Learning the measurement results from Alice via the classical communication channel, Bob applies a corresponding unitary transformation on the other part of the entangled state ρ, so as to transform the state of this part to the unknown state $|\phi\rangle$. In the following we investigate this traditional faithful teleportation with arbitrary dimensional bipartite entangled state ρ.

We first consider the case that ρ in $\mathbb{C}^m \otimes \mathbb{C}^n$ $(m, n \geq d)$.

Theorem 1 *If a mixed state $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, $p_i \geq 0$, $\sum_{i=1}^{k} p_i = 1$, is an ideal resource for faithful teleportation, then every pure state $|\psi_i\rangle$ is the ideal resource for faithful teleportation.*

Proof. Let $\{|\tilde{\psi}_j\rangle\}_{j=1}^{r}$ be an orthonormal basis in $\mathbb{C}^r$, $\sum_{j=1}^{r} |\tilde{\psi}_j\rangle\langle\tilde{\psi}_j| = I_r$, where I_r is the $r \times r$ identity matrix. Assume Alice makes a complete projective measurement $\{|\tilde{\psi}_j\rangle\langle\tilde{\psi}_j|\}_{j=1}^{r}$ on the initial state

$|\phi\rangle\langle\phi| \otimes \rho$, where $|\phi\rangle$ is unknown state to be teleported. Then Bob applies the unitary operation $U^{(j)\dagger}$ on his part with respect to Alice's measurement result j. A faithful teleportation implies that

$$|\phi\rangle\langle\phi| \otimes \rho = \sum_{i=1}^{k} p_i |\phi\rangle\langle\phi| \otimes |\psi_i\rangle\langle\psi_i| = \sum_{j,j'=1}^{r} q_j |\tilde{\psi}_j\rangle\langle\tilde{\psi}_{j'}| \otimes U^{(j)}|\phi\rangle\langle\phi|U^{(j')\dagger} \tag{1}$$

with $\sum_{j=1}^{r} q_j = 1$, $q_j \geq 0$, $j = 1, \cdots, r$. After Alice's complete projective measurement, from Eq. (1) we obtain

$$\sum_{i=1}^{k} p_i \langle\tilde{\psi}_j|(|\phi\rangle\langle\phi| \otimes |\psi_i\rangle\langle\psi_i|)|\tilde{\psi}_j\rangle = q_j U^{(j)}|\phi\rangle\langle\phi|U^{(j)\dagger} \tag{2}$$

and

$$\langle\tilde{\psi}_j|(|\phi\rangle\langle\phi| \otimes |\psi_i\rangle\langle\psi_i|)|\tilde{\psi}_j\rangle = f_{ij} U^{(j)}|\phi\rangle\langle\phi|U^{(j)\dagger}, \tag{3}$$

with $\sum_{i=1}^{k} \sum_{j=1}^{r} f_{ij} = 1$, $f_{ij} \geq 0$, $i = 1, \cdots, k$, $j = 1, \cdots, r$. From Eq. (3) we have

$$\sum_{j=1}^{r} \langle\tilde{\psi}_j|(|\phi\rangle\langle\phi| \otimes |\psi_i\rangle\langle\psi_i|)|\tilde{\psi}_j\rangle = \sum_{j=1}^{r} f_{ij} U^{(j)}|\phi\rangle\langle\phi|U^{(j)\dagger} \tag{4}$$

for $i = 1, \cdots, k$. Due to the completeness of the projectors $\{|\tilde{\psi}_j\rangle\langle\tilde{\psi}_j|\}$, f_{ij} satisfies $\sum_{j=1}^{r} f_{ij} = 1$ for each i. Therefore every pure state $|\psi_i\rangle$ must be an ideal resource for faithful teleportation. $\square$

From the theorem one has the following necessary condition of mixed states for faithful teleportation.

Corollary 2 *If mixed state ρ is the ideal resource for faithful teleportation, then its eigenstates must be all ideal resource for faithful teleportation.*

Utilizing the necessary conditions of mixed states, we consider now which kind of states in $\mathbb{C}^m \otimes \mathbb{C}^n$ can be used as entangled resource for faithful teleportation of an unknown state $|\phi\rangle$ in $\mathbb{C}^d$. We systematically study the problem in four cases.

Case i). *$n = d$, pure states*

Since any pure state $|\psi\rangle$ in $\mathbb{C}^m \otimes \mathbb{C}^d$ can be transformed into some pure state in $\mathbb{C}^d \otimes \mathbb{C}^d$ under local unitary transformations, we only need to consider pure states $|\psi\rangle$ in $\mathbb{C}^d \otimes \mathbb{C}^d$. In this case, it has been shown that only maximally entangled pure states can be used for faithful teleportation in $\mathbb{C}^d \otimes \mathbb{C}^d$, if the Bell measurements are applied by Alice [1–3]. Here we give an alternative proof of this result for consistency and the use for the rest of this paper. In addition, from the proof we show that, to fulfill faithful teleportation, Alice's measurements must be projectors of maximally entangled pure states.

Theorem 3 *A pure state $|\psi\rangle$ in $\mathbb{C}^d \otimes \mathbb{C}^d$ is an ideal resource for faithful teleportation of $|\phi\rangle = \sum_{i=1}^{d} \alpha_i |i\rangle$, if and only if $|\psi\rangle$ is maximally entangled. Moreover, Alice's measurements must be all projectors of maximally entangled states.*

Proof. Let $|\psi\rangle = \sum_{i,j=1}^{d} a_{ij}|ij\rangle$ be the entangled pure state shared by Alice and Bob. To teleport the unknown state $|\phi\rangle$, Alice carries out complete measurements $\{|\psi_{st}\rangle\langle\psi_{st}|\}_{s,t=1}^{d}$, where $|\psi_{st}\rangle = \sum_{p,q=1}^{d} U_{pq,st}|pq\rangle$ satisfying $\langle\psi_{s',t'}|\psi_{s,t}\rangle = \sum_{p,q=1}^{d} U_{pq,st}U^*_{pq,s't'} = \delta_{s,s'}\delta_{t,t'}$. One has

$$
\begin{aligned}
|\phi\rangle|\psi\rangle &= \sum_{i,j,k=1}^{d} \alpha_i a_{jk}|ijk\rangle = \sum_{s,t=1}^{d}\sum_{i,j,k=1}^{d} \alpha_i a_{jk}|\psi_{st}\rangle\langle\psi_{st}|ijk\rangle \\
&= \sum_{s,t=1}^{d} |\psi_{st}\rangle\Big(\sum_{i,j,k=1}^{d} U^*_{ij,st}\alpha_i a_{jk}|k\rangle\Big) = \sum_{s,t=1}^{d} |\psi_{st}\rangle A^T V^\dagger_{st}|\phi\rangle,
\end{aligned}
\tag{5}
$$

where $(V_{st})_{ij} = (U_{ij,st})$ and $(A)_{ij} = (a_{ij})$ are the coefficient matrices of $|\psi_{st}\rangle$ and $|\psi\rangle$ respectively. If $|\psi\rangle$ is an ideal resource for faithful teleportation, then $A^T V^\dagger_{st}$ should be unitary up to a constant factor:

$$
A^T V^\dagger_{st} = c_{st} W_{st}, \tag{6}
$$

for some unitary matrix W_{st} and $0 \le c_{st} \le 1$, $s, t = 1, \cdots, d$,

$$
\sum_{s,t=1}^{d} |c_{st}|^2 = 1. \tag{7}
$$

Let $A^T = U_1\Lambda_1 V_1$ and $V^\dagger_{st} = U_{2,st}\Lambda_{2,st}V_{2,st}$ be the singular decompositions of A^T and $V^\dagger_{st}$ respectively, where $U_1, V_1, U_{2,st}, V_{2,st}$ are unitary matrices, $\Lambda_1 = diag(\lambda_1, \cdots, \lambda_d)$, $\Lambda_{2,st} = diag(\mu_{1,st}, \cdots, \mu_{d,st})$, λ_i and $\mu_{i,st}$ are the corresponding singular values. Due to the normality, $trAA^\dagger = trV_{st}V^\dagger_{st} = 1$, one gets

$$
\sum_{i=1}^{d} |\lambda_i|^2 = 1, \tag{8}
$$

$$
\sum_{i=1}^{d} |\mu_{i,st}|^2 = 1. \tag{9}
$$

Then from Eq. (6) we have

$$
c^2_{st} I_d = A^T V^\dagger_{st} V_{st} A^{T\dagger} = U_1\Lambda_1 V_1 U_{2,st}\Lambda_{2,st}\Lambda^\dagger_{2,st}U^\dagger_{2,st}V^\dagger_1\Lambda^\dagger_1 U^\dagger_1 \tag{10}
$$

and

$$
c^2_{st}(\Lambda_1\Lambda^\dagger_1)^{-1} = V_1 U_{2,st}\Lambda_{2,st}\Lambda^\dagger_{2,st}U^\dagger_{2,st}V^\dagger_1, \tag{11}
$$

which give rise to

$$|\lambda_i|^{-2} = \frac{1}{c_{st}^2}|\mu_{i,st}|^2 \tag{12}$$

and

$$|\lambda_i|^2 = \frac{c_{st}^2}{|\mu_{i,st}|^2} \tag{13}$$

by reordering $\{\lambda_i\}$ and $\{\mu_{i,st}\}$. From Eq. (8) and Eq. (12) we have

$$\frac{1}{c_{st}^2}\sum_{i=1}^{d}|\mu_{i,st}|^2 = \sum_{i=1}^{d}|\lambda_i|^{-2} \leq d^2.$$

Using Eq. (9) one has $c_{st}^2 \geq 1/d^2$ for $s, t = 1, \cdots, d$, which results in

$$c_{st}^2 = \frac{1}{d^2} \tag{14}$$

by taking into account Eq. (7). Inserting Eq. (14) into Eq. (13) and using Eq. (8) we get

$$\sum_{i=1}^{d}\frac{1}{|\mu_{i,st}|^2} = d^2. \tag{15}$$

Hence in terms of Eq. (9), Eq. (13) and Eq. (15) we obtain

$$|\lambda_i|^2 = |\mu_{i,st}|^2 = \frac{1}{d}, \quad i, s, t = 1, \cdots, d, \tag{16}$$

which are just the square of the Schmidt coefficients of $|\psi\rangle$ and $|\psi_{st}\rangle$ respectively. Therefore $A^T = \frac{1}{\sqrt{d}}\tilde{U}$ and $V_{st} = \frac{1}{\sqrt{d}}\tilde{V}_{st}$ for some unitary matrices $\tilde{U}$ and $\tilde{V}_{st}$. As a result, the shared entangled state $|\psi\rangle$ and the state $|\psi_{st}\rangle$ in the projective measurements $\{|\psi_{st}\rangle\langle\psi_{st}|\}$ must be all maximally entangled ones. At last, the initial state can be expressed as

$$|\phi\rangle|\psi\rangle = \frac{1}{d}\sum_{s,t=1}^{d}|\psi_{st}\rangle\tilde{U}_{st}|\phi\rangle, \tag{17}$$

where $\tilde{U}_{st} = dA^T V_{st}^\dagger$ is determined by the shared state $|\psi\rangle$ and the projective measurement operators $|\psi_{st}\rangle\langle\psi_{st}|$. To carry out the teleportation, Alice measures her particles by d^2 orthonormal projectors, and informs Bob the measurement results. Each result appears in her measurements with probability $\frac{1}{d^2}$. According to Alice's measurement result st, Bob fulfills faithful teleportation by applying unitary operation $\tilde{U}_{st}^\dagger$ on his part of the entangled resource.

Conversely, if the shared entangled state is maximally entangled, it is straightforward to prove that faithful teleportation can be carried out. A maximally entangled state can be generally expressed as $U_1 \otimes U_2|\psi\rangle$, where $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|ii\rangle$, U_1 and U_2 are unitary matrices. Taking $|\psi\rangle$ as the entangled resource to teleport $|\phi\rangle = \sum_{i=1}^{d}\alpha_i|i\rangle$, one has

$$|\phi\rangle \otimes |\psi\rangle = \frac{1}{d}\sum_{s,t=1}^{d}|\psi_{st}\rangle \otimes U_{st}|\phi\rangle,$$

where $|\psi_{st}\rangle = \frac{1}{\sqrt{d}}U_{st}^{\dagger} \otimes I|\psi\rangle$, $s, t = 1, 2, \cdots, d$. Here $\{U_{st}\}$ is the basis of the unitary operators satisfying $tr(U_{st}U_{s't'}^{\dagger}) = d\delta_{ss'}\delta_{tt'}$ and $tr(U_{st}U_{st}^{\dagger}) = I_d$. For instance, one could choose $U_{st} = h^t g^s$ with $d \times d$ matrices h and g such that $h|j\rangle = |(j+1) \mod d\rangle$, $g|j\rangle = \omega^j|j\rangle$, $\omega = \exp\{-2i\pi/d\}$, $s, t = 1, 2, \cdots, d$, as the basis of the unitary operators to perform the faithful teleportation. Faithful teleportation with other maximally entangled pure states can be similarly analyzed. $\square$

Case ii). *$n = d$, mixed states*

Theorem 4 *A mixed state ρ in $\mathbb{C}^m \otimes \mathbb{C}^d$ with rank k can be used as the entangled resource for faithful teleportation of $|\phi\rangle = \sum_{i=1}^{d}\alpha_i|i\rangle$, if and only if ρ is the mixed maximally entangled state [9]: $\rho = \sum_{x=1}^{k} p_x|\psi_x\rangle\langle\psi_x|$, where $|\psi_x\rangle$ is maximally entangled in $H_x \otimes \mathbb{C}^d$, $\dim H_x = d$, $x = 1, \cdots, k$, $\{H_x\}$ are complex vector spaces that are orthogonal to each other.*

One of the maximally entangled state in $\mathbb{C}^m \otimes \mathbb{C}^d$ $(m \geq d)$ is of the form, $|\phi\rangle=\sum_{i=1}^{d}\frac{1}{\sqrt{d}}|ii\rangle$. Quantified by a certain entanglement measure, a mixed maximally entangled state has the same degree of entanglement as this pure maximally entangled state. This fact holds true for any entanglement measures that does not increase under local operations and classical communications in literature [9].

Proof. It has been proved that the mixed maximally entangled state [9] can be used to fulfill faithful teleportation. Now we prove the converse. Suppose the mixed state ρ with rank k in $\mathbb{C}^m \otimes \mathbb{C}^d$ is the ideal resource for teleportation. From corollary 2 we have that the k orthogonal eigenstates $\{|\psi_i\rangle\}$ of ρ with respect to nonzero eigenvalues are all maximally entangled. In fact, they can be constructed in the following way. We assume, without loss of generality, $|\psi_1\rangle$ be maximally entangled in $H_1 \otimes \mathbb{C}^d$ with $H_1 = \mathbb{C}^d = Span\{|1\rangle, \cdots, |d\rangle\}$, where $\{|i\rangle\}_{i=1}^{d}$ is an orthonormal basis of $\mathbb{C}^d$. From Eq. (4) one has

$$\sum_{j=1}^{d^2}\langle\tilde{\psi}_j|(|\phi\rangle\langle\phi| \otimes |\psi_1\rangle\langle\psi_1|)|\tilde{\psi}_j\rangle = \sum_{j=1}^{d^2} f_{1j}U^{(j)}|\phi\rangle\langle\phi|U^{(j)\dagger}, \tag{18}$$

where $f_{1j} = 1/d^2$ for $j = 1, \cdots, d^2$, and $f_{1j} = 0$ for $j > d^2$. Here $\{|\tilde{\psi}_j\rangle\}_{j=1}^{d^2}$ are maximally entangled states constituting an orthonormal basis in $\mathbb{C}^d \otimes H_1$. Similarly state $|\psi_2\rangle$ satisfies

$$\sum_{j=d^2+1}^{2d^2} \langle\tilde{\psi}_j|(|\phi\rangle\langle\phi| \otimes |\psi_2\rangle\langle\psi_2|)|\tilde{\psi}_j\rangle = \sum_{j=d^2+1}^{2d^2} f_{2j} U^{(j)}|\phi\rangle\langle\phi|U^{(j)\dagger} \tag{19}$$

with $f_{2j} = 1/d^2$ for $j = d^2+1, \cdots, 2d^2$, and $f_{2j} = 0$ otherwise. $\{|\tilde{\psi}_j\rangle\}_{j=d^2+1}^{2d^2}$ are maximally entangled and constitute an orthonormal basis in $\mathbb{C}^d \otimes H_2$, $\dim H_2 = d$. Hence $|\psi_2\rangle$ is maximally entangled in $H_2 \otimes \mathbb{C}^d$. From the orthogonality of $\{|\tilde{\psi}_j\rangle\}_{j=1}^{d^2}$ and $\{|\tilde{\psi}_j\rangle\}_{j=d^2+1}^{2d^2}$, we know that H_2 is orthogonal to H_1. Other eigenstates $|\psi_x\rangle$, $2 \le x \le k$, can be treated similarly,

$$\sum_{j=(x-1)d^2+1}^{xd^2} \langle\tilde{\psi}_j|(|\phi\rangle\langle\phi| \otimes |\psi_x\rangle\langle\psi_x|)|\tilde{\psi}_j\rangle = \sum_{j=(x-1)d^2+1}^{xd^2} f_{xj} U^{(j)}|\phi\rangle\langle\phi|U^{(j)\dagger}. \tag{20}$$

$\{|\tilde{\psi}_j\rangle\}_{j=(x-1)d^2+1}^{xd^2}$ are maximally entangled and constitute the orthonormal basis in $\mathbb{C}^d \otimes H_x$, where $\{H_x\}$ are orthogonal to each other, $\dim H_x = d$ for $x = 1, \cdots, k$. Hence $|\psi_x\rangle$ is maximally entangled in $H_x \otimes \mathbb{C}^d$. From the above analysis, we have $m \ge kd$ with k the rank of ρ. The probability of the outcomes of each measurement depends on the eigenvalues p_i, $i = 1, \cdots, k$. Therefore a mixed state ρ in $\mathbb{C}^m \otimes \mathbb{C}^d$ that can be used for faithful teleportation of $|\phi\rangle$ in $\mathbb{C}^d$ must be a mixed maximally entangled state. $\square$

As an example, we consider the faithful teleportation of $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ by using a mixed maximally entangled state ρ_0 in $\mathbb{C}^4 \otimes \mathbb{C}^2$, $\rho_0 = \frac{1}{2}|\psi_1^+\rangle\langle\psi_1^+| + \frac{1}{2}|\psi_2^+\rangle\langle\psi_2^+|$ with $|\psi_1^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $|\psi_2^+\rangle = \frac{1}{\sqrt{2}}(|20\rangle + |31\rangle)$. By straightforward calculations one has

$$\begin{aligned} |\phi\rangle\langle\phi| \otimes \rho_0 = \quad & \frac{1}{8}(|\psi_1^+\rangle \otimes |\phi\rangle + |\psi_1^-\rangle \otimes \sigma_3|\phi\rangle + |\phi_1^+\rangle \otimes \sigma_1|\phi\rangle + |\phi_1^-\rangle \otimes \sigma_2|\phi\rangle) \\ & (\langle\psi_1^+| \otimes \langle\phi| + \langle\psi_1^-| \otimes \sigma_3\langle\phi| + \langle\phi_1^+| \otimes \sigma_1\langle\phi| - \langle\phi_1^-| \otimes \sigma_2\langle\phi|) \\ & + \frac{1}{8}(|\psi_2^+\rangle \otimes |\phi\rangle + |\psi_2^-\rangle \otimes \sigma_3|\phi\rangle + |\phi_2^+\rangle \otimes \sigma_1|\phi\rangle + |\phi_2^-\rangle \otimes \sigma_2|\phi\rangle) \\ & (\langle\psi_2^+| \otimes \langle\phi| + \langle\psi_2^-| \otimes \sigma_3\langle\phi| + \langle\phi_2^+| \otimes \sigma_1\langle\phi| - \langle\phi_2^-| \otimes \sigma_2\langle\phi|), \end{aligned}$$

where $|\psi_1^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$, $|\phi_1^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$, $|\psi_2^-\rangle = \frac{1}{\sqrt{2}}(|20\rangle - |31\rangle)$ and $|\phi_2^\pm\rangle = \frac{1}{\sqrt{2}}(|21\rangle \pm |30\rangle)$ are maximally entangled states, $\sigma_1 = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\sigma_2 = -|0\rangle\langle 1| + |1\rangle\langle 0|$ and $\sigma_3 = |0\rangle\langle 0| - |1\rangle\langle 1|$ are Pauli matrices. Above relation implies that the faithful teleportation can be carried out with the mixed maximally entangled state ρ_0.

In fact, this mixed state ρ_0 could be evolved from a maximally entangled pure state $|\psi_0\rangle = \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |33\rangle)$. If the second part of the pure state $|\psi_0\rangle$ undergoes a noisy channel with

Kraus operators $A_1 = |0\rangle\langle 0| + |1\rangle\langle 1|$ and $A_2 = |2\rangle\langle 2| + |3\rangle\langle 3|$, $|\psi_0\rangle$ becomes $\rho_0 = \Lambda \otimes I(|\psi_0\rangle\langle\psi_0|)$. Usually under noisy channel a maximally entangled pure state becomes a mixed one which can be no longer used for faithful teleportation. Our example shows that this channel does not influence the capability of the input state for faithful teleportation of one qubit state.

Case iii). *$m = d$, pure or mixed states*

Theorem 5 *An entangled state ρ in $\mathbb{C}^d \otimes \mathbb{C}^n$ can be used for faithful teleportation of $|\phi\rangle = \sum_{i=1}^{d} \alpha_i |i\rangle$, if and only if it is a maximally entangled pure state in $\mathbb{C}^d \otimes \mathbb{C}^d$.*

Proof. If ρ with rank k is able to teleport $|\phi\rangle$ faithfully, from corollary 2 its eigenstates should be all ideal resources for teleportation. Hence according to the proof of theorem 4, ρ's eigenvectors associated with the first subsystem must be orthogonal to each other. Namely the dimension of the first subsystem of ρ should be kd. Since the dimension of the first subsystem of ρ is d, we have $k = 1$ and ρ is a maximally entangled pure state. $\square$

In the above discussions we have used the traditional teleportation protocol: Alice makes a projective measurement first, Bob applies unitary operations correspondingly then. In fact, if the entangled resource is in $\mathbb{C}^d \otimes \mathbb{C}^n$, Bob performs a projective measurement first, then more entangled states may be served as ideal resources for faithful teleportation.

Theorem 6 *State ρ in $\mathbb{C}^d \otimes \mathbb{C}^n$ can be used for faithful teleportation of state $|\phi\rangle = \sum_{i=1}^{d} \alpha_i |i\rangle$, if and only if ρ is the mixed maximally entangled state: $\rho = \sum_{x=1}^{k} p_x |\psi_x\rangle\langle\psi_x|$, where $|\psi_x\rangle$ is maximally entangled in $\mathbb{C}^d \otimes H_x$, $\dim H_x = d$, $x = 1, \cdots, k$, $n \geq kd$, $\{H_x\}$ are complex vector spaces that are orthogonal to each other.*

Proof. Let Bob perform a complete measurement first. To carry out faithful teleportation, from theorem 5 the post-measurement state in $\mathbb{C}^d \otimes \mathbb{C}^{n'}$, $(n' < n)$, must be a maximally entangled pure state in $\mathbb{C}^d \otimes \mathbb{C}^d$. Since local operations do not increase entanglement, the initial state ρ with rank k must be a maximally entangled one in $\mathbb{C}^d \otimes \mathbb{C}^d$. This implies that $n \geq kd$ and $\rho = \sum_{x=1}^{k} p_x |\psi_x\rangle\langle\psi_x|$ with $|\psi_x\rangle$ the maximally entangled state in $\mathbb{C}^d \otimes H_x$, $\dim H_x = d$, $x = 1, \cdots, k$, $\{H_x\}$ are orthogonal to each other.

On the other hand, if Bob's measurement can project the maximally entangled states in $\mathbb{C}^d \otimes \mathbb{C}^n$ to be the above maximally entangled states in $\mathbb{C}^d \otimes H_x$, they can be used for faithful teleportation obviously. Therefore, state ρ in $\mathbb{C}^d \otimes \mathbb{C}^n$ can be used for faithful teleportation if and only if it is the mixed maximally entangled and $n \geq kd$. $\square$

Case iv). $m, n > d$, *pure or mixed states*

Concerning pure states in $m, n > d$, we introduce a class of states:

$$|\psi\rangle = c_1|\psi_1\rangle + \cdots + c_l|\psi_l\rangle, \tag{21}$$

where $|\psi_p\rangle \in H_p^A \otimes H_p^B$ is a maximally entangled state, $\{H_p^A\}_{p=1}^l$ are complex vector spaces that are orthogonal to each other, $\dim H_p^A = \dim H_p^B = n_p \geq d$ for $p = 1, \cdots, l$, $\sum_{p=1}^l n_p \leq \min\{m,\ n\}$ and $\sum_{i=1}^l |c_i|^2 = 1$. Without loss of generality, we assume Alice's measurements are given by the projectors $\{|\tilde{\psi}_{st,p}\rangle\langle\tilde{\psi}_{st,p}|\}$, $s = 1, \cdots, d$, $t = 1, \cdots, n_p$, $p = 1, \cdots, l$. Here for $p = 1, \cdots, l$, $\{|\tilde{\psi}_{st,p}\rangle\langle\tilde{\psi}_{st,p}|\}$ are projectors onto $\mathbb{C}^d \otimes H_p^A$. And $\{|\tilde{\psi}_{st,p}\rangle\}$ are maximally entangled states constituting an orthonormal basis in $\mathbb{C}^d \otimes H_p^A$. We have

$$|\phi\rangle|\psi\rangle = \frac{1}{\sqrt{dn_p}} \sum_{p=1}^{l} \sum_{s=1}^{d} \sum_{t=1}^{n_p} c_p |\tilde{\psi}_{st,p}\rangle \tilde{U}_{st,p}|\phi\rangle, \tag{22}$$

where the unitary matrix $\tilde{U}_{st,p}$ depends on the shared resource state $|\psi\rangle$ and the measurement operator $|\tilde{\psi}_{st,p}\rangle\langle\tilde{\psi}_{st,p}|$. The probability of Alice's each measurement outcome is $\frac{|c_p|^2}{dn_p}$.

For example, let us consider the teleportation of a qubit state $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ with non-maximally entangled pure state $|\psi\rangle = \sqrt{a}|\eta\rangle + \sqrt{1-a}|\xi\rangle$, with $|\eta\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $|\xi\rangle = \frac{1}{\sqrt{3}}(|22\rangle + |33\rangle + |44\rangle)$. We have

$$\begin{aligned}
|\phi\rangle \otimes |\psi\rangle \;=\; & \frac{\sqrt{a}}{2}\Big(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes U_1|\phi\rangle + \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \otimes U_2|\phi\rangle \\
& + \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \otimes U_3|\phi\rangle + \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \otimes U_4|\phi\rangle\Big) \\
& + \sqrt{\frac{1-a}{6}}\Big(\frac{1}{\sqrt{2}}(|02\rangle + |13\rangle) \otimes V_1|\phi\rangle + \frac{1}{\sqrt{2}}(|02\rangle - |13\rangle) \otimes V_2|\phi\rangle \\
& + \frac{1}{\sqrt{2}}(|03\rangle + |14\rangle) \otimes V_3|\phi\rangle + \frac{1}{\sqrt{2}}(|03\rangle - |14\rangle) \otimes V_4|\phi\rangle \\
& + \frac{1}{\sqrt{2}}(|04\rangle + |12\rangle) \otimes V_5|\phi\rangle + \frac{1}{\sqrt{2}}(|04\rangle - |12\rangle) \otimes V_6|\phi\rangle\Big),
\end{aligned}$$

where $U_{1,2} = |0\rangle\langle 0| \pm |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| + |4\rangle\langle 4|$, $U_{3,4} = |1\rangle\langle 0| \pm |0\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| + |4\rangle\langle 4|$, $V_{1,2} = |0\rangle\langle 2| + |2\rangle\langle 0| \pm |1\rangle\langle 3| \pm |3\rangle\langle 1| + |4\rangle\langle 4|$, $V_{3,4} = |0\rangle\langle 3| + |3\rangle\langle 0| \pm |1\rangle\langle 4| \pm |4\rangle\langle 1| + |2\rangle\langle 2|$, $V_{5,6} = |0\rangle\langle 4| + |4\rangle\langle 0| \pm |1\rangle\langle 2| \pm |2\rangle\langle 1| + |3\rangle\langle 3|$. Obviously with respect to the Alice's measurement results, faithful teleportation can be realized by applying the corresponding unitary transformations U_i and V_j, $i = 1, ..., 4,\ j = 1, ..., 6$, on the Bob's part.

For mixed states ρ in $\mathbb{C}^m \otimes \mathbb{C}^n$ if all its eigenstates belong to the class Eq. (21) and any superpositions of these eigenstates belong to the class too, then ρ can be used for faithful teleportation.

For instance, $\rho = p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2|$ with $|\psi_1\rangle = \frac{1}{2}(|00\rangle + |11\rangle) + \frac{1}{\sqrt{6}}(|22\rangle + |33\rangle + |44\rangle)$, $|\psi_2\rangle = \frac{1}{2}(|00\rangle + |11\rangle) + \frac{1}{2}(|52\rangle + |63\rangle)$, $p_1 + p_2 = 1$, can be used for perfect teleportation of one qubit state.

Remark. In [10] multipartite entangled states are used in faithful teleportation of d-qubit state. If we treat such teleportation as the teleportation of pure states in $\mathbb{C}^{2^d}$, then it is easy to verify that the shared resources belong to the class of states Eq. (21).

In the following we discuss the relations between the degree of entanglement of the resource state ρ and faithful teleportation. Here we use the well-known entanglement measure, entanglement of formation [11] to characterize the entanglement. For a pure bipartite state $|\psi\rangle_{AB}$, the entanglement of formation is defined as the von Neumann entropy of either of the two subsystems A and B: $E(|\psi\rangle_{AB}) = -tr(\rho_{A(B)} \log_2 \rho_{A(B)})$, where $\rho_{A(B)} = tr_{B(A)}(|\psi\rangle_{AB}\langle\psi|)$ are the reduced density matrices. For mixed state ρ with pure state decompositions $\rho = \sum_i p_i|\phi_i\rangle\langle\phi_i|$, $\sum_i p_i = 1$, the entanglement of formation is defined as the average entanglement of the pure states in the decomposition, minimized over all possible pure state decompositions of ρ: $E(\rho) = \inf \sum_i p_i E(|\phi_i\rangle)$.

It can be verified that either the maximally entangled pure state $|\psi\rangle$ in $\mathbb{C}^d \otimes \mathbb{C}^d$, or the mixed maximally entangled state ρ in $\mathbb{C}^m \otimes \mathbb{C}^d$, the entanglement of formation is $E = \log d$. And entangled states with entanglement of formation $E = \log d$ in these vector spaces must be pure or mixed maximally entangled [9]. Hence one has that the states in $\mathbb{C}^m \otimes \mathbb{C}^d$ ($m \geq d$) can be used for faithful teleportation if and only if their entanglement of formation is $\log d$. If Bob is allowed to apply the measurements before the traditional teleportation protocol, the states in $\mathbb{C}^d \otimes \mathbb{C}^n$ ($n \geq d$) can be used for faithful teleportation if and only if their entanglement of formation is $\log d$. For states ρ in $\mathbb{C}^m \otimes \mathbb{C}^n$ with $m, n > d$, the entanglement of formation of ρ presented in this paper for faithful teleportation is larger than or equal to $\log d$. One may conjecture that the entanglement of formation for all ideal entangled resources is not less than $\log d$, giving rise to another necessary condition for quantum states to be used for faithful teleportation.

III. CONLUSIONS

We have investigated systematically the necessary conditions of entangled resource state $\mathbb{C}^m \otimes \mathbb{C}^n$ ($m, n \geq d$) for faithful teleportation of $|\phi\rangle$ in $\mathbb{C}^d$. It has been shown that for $n = d$, ρ can be used for faithful teleportation if and only if it is maximally entangled. For $m = d$, ρ can be used for faithful teleportation if and only if it is a maximally entangled pure state. For $m, n > d$, we present

a class of pure and mixed states that can be used for faithful teleportation. From the point of view of experimental implementation of quantum teleportation [12], our results may help to understand the character of faithful teleportation and to facilitate the experimental implementations.

Acknowledgments M. J. Zhao thanks Max-Planck-Institute for Mathematics in the Sciences, Leipzig for the hospitality. This work is supported by the NSFC 10875081, NSFC 10871227, KZ200810028013, PHR201007107 and NSF of Beijing 1092008.

---

[1] Bennett C H, Brassard G, Crépeau C, Jozsa R, Peres A and Woottes W K 1993 *Phys. Rev. Lett.* **70** 1895

[2] Horodecki M, Horodecki P and Horodecki R 1999 *Phys. Rev. A* **60** 1888

[3] Albeverio S, Fei S M and Yang W L 2002 *Phys. Rev. A* **66** 012301

[4] Karlsson A and Bourennane M 1998 *Phys. Rev. A* **58** 4394

[5] Agrawal P and Pati A 2006 *Phys. Rev. A* **74** 062320

[6] Muralidharan S and Panigrahi P K 2008 *Phys. Rev. A* **77** 032321

[7] Rigolin G 2005 *Phys. Rev. A* **71** 032303

[8] Yeo Y and Chua W K 2006 *Phys. Rev. Lett.* **96** 060502

[9] Li Z G, Zhao M J, Fei S M, Fan H and Liu W M 2009 *arXiv: quant-ph/* 0906.5445

[10] Cheung C Y and Zhang Z J 2009 *Phys. Rev. A* **80** 022327

[11] Wootters W K 1998 *Phys. Rev. Lett.* **80** 2245

[12] Bouwmeester D, Pan J W, Daniell M, Weinfurter H and Zeilinger A 1997 *Nature (London)* **390** 575
Boschi D, Branca S, Martini F De, Hardy L and Popescu S 1998 *Phys. Rev. Lett.* **80** 1121